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ESTIMATE OF THE GUARANTEED VALUE IN A NON-LINEAR DIFFERENTIAL GAME OF APPROACH*

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A non-linear differential game of approach is considered. An estimate of the guaranteed value function is derived. The formalization of the problem follows /1, 2/, using constructions based on the conjugate derivative of locally Lipschitzian functions and the Hamiltonian function characterizing the dynamics of the non-linear systems /3-7/. An auxiliary linear differential game of approach is introduced with a known u-stable guaranteed value function. The construction are based on the results of /8-10/. The Hamiltonians of the original non-linear system and the auxiliary linear system are not assumed to be related in any special way. An example is considered in which the proposed approach is used to construct a bound on the guaranteed value function. The problem is related to the previous studies /11-16/.

1. ***The equations of motion and reward functional.*** We consider a controlled system described by the equation

$$x^{\cdot} = f(t, x, u, v),\ t \in I,\ x \in R^n,\ u \in P,\ v \in Q \tag{1.1}$$

where $I = [t_0, \theta]$, $f: I \times R^n \times P \times Q \to R^n$, P and Q are compact sets in R^P and R^Q respectively. The function f is assumed to be continuous and satisfies the Lipschitz condition

$$\sup_{(t,\, x^{(i)},\, u,\, v)} \| f(t, x^{(1)}, u, v) - f(t, x^{(2)}, u, v) \| \, \| x^{(1)} - x^{(2)} \|^{-1} < \infty$$

$$x \in G, (x^{(1)} \neq x^{(2)})$$

where G is a bounded region in R^n containing all possible motions of the system (1.1) that leave the given compact set.

Let $\|x\| = \langle x, x\rangle^{1/2}$, where the symbol $\langle,\rangle$ denotes the scalar product. The motion of system (1.1) is considered in a fixed interval, $t \in I$.

The reward functional γ in this game has the form

$$\gamma(x(\cdot)) = \sigma(x(\theta)) \tag{1.2}$$

where the function $\sigma: R^n \to R^1$ is Lipschitzian.

The formalization of the differential game (1.1), (1.2) is based on /1, 2/. Let ω_0: $I \times R^n \to R^1$ be the value function of the game. We know /2/ that it satisfies the Lipschitz condition

$$\sup | \omega_0(t^{(1)}, x^{(1)}) - \omega_0(t^{(2)}, x^{(2)}) | \times (| t^{(1)} - t^{(2)} | + \| x^{(1)} - x^{(2)} \|)^{-1} < \infty$$

$$(t^{(i)}, x^{(i)}) \in I \times G\ (i = 1, 2),\ | t^{(1)} - t^{(2)} | + \| x^{(1)} - x^{(2)} \| > 0$$

By definition, we have

$$\omega^{\circ}(\theta, x) = \sigma(x) \tag{1.3}$$

2. ***Auxiliary differential games.*** We introduce an axuiliary system described by the linear equation

$$x^{\cdot} = A(t)x + B(t)u_1 + C(t)v_1,\ t \in I,\ x \in R^n,\ u_1 \in P_1,\ v_1 \in Q_1 \tag{2.1}$$

Here P_1 and Q_1 are compact sets in R^P and R^Q, respectively. The matrix functions $A(t)$, $B(t)$ and $C(t)$ are assumed to be continuous.

Assume that some u-stable function $\omega^{\circ}(t, x)$: $I \times R^n \to R^1$ is defined for the auxiliary system (2.1) (in particular, the value function of the game), satisfying the relationship

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$$\omega^\circ(\theta, x) = \sigma(x) \tag{2.2}$$

We introduce another auxiliary game. To this end, we define the function $\varphi(t, x)$ by the expression

$$\varphi(t, x) = \max_{s \in S^*} |H(t, x, s) - H_1(t, x s)|, \quad S^* = \{s \subset R^n : \|s\| \leqslant 1\} \tag{2.3}$$

Here $H(t, x, s)$ and $H_1(t, x, s)$ are the Hamiltonians of the original system (1.1) and the auxiliary system (2.1), respectively.

Consider the auxiliary differential game described by the equation

$$x^{\cdot} = A(t)x + B(t)u_1 + C(t)v_1 + \varphi(t, x)v, \ t \in I, \ x \in R^n \ u_1 \in P_1 \tag{2.4}$$
$$v_1 \in Q_1$$

where $A(t)$, $B(t)$, $C(t)$ are the same matrices as in Eq.(2.1), the function $\varphi(t, x)$ is defined by (3.1), and v is an n-dimensional vector, $\|v\| \leqslant 1$.

We assume that the constraints on the controls u_1, v_1 and the reward functional in the differential game described by Eq.(2.4) are the same as in (2.1) and (1.2), respectively.

Remark 1. The Hamiltonian function of the system (3.2) given by

$$H_1(t, x, s) + \varphi(t, x)\|s\| \tag{2.5}$$

and the Hamiltonian function of system (1.1) (using expression (2.3)) are related by the inequality

$$H_1(t, x, s) + \varphi(t, x)\|s\| \geqslant H(t, x, s), \quad (t, x, s) \in I \times G \times R^n \tag{2.6}$$

3. ***Statement of the problem.*** It is required to construct a u-stable function $\omega_2^1(t, x)$: $I \times R^n \to R^1$ for the original non-linear differential game (1.1), (1.2) that satisfies the Lipschitz condition and the boundary condition

$$\omega_2^1(\theta, x) = \sigma(x)$$

We will seek the function $\omega_2^1(t, x)$ in the form

$$\omega_2^1(t, x) = \omega^\circ(t, x) + \Delta\omega^1(t, x) + \Delta\psi(t) \tag{3.1}$$

Here $\omega^\circ(t, x)$ is the value function of the auxiliary linear game (2.1), (2.2), and $\Delta\omega^1(t, x)$ and $\Delta\psi(t)$ are some differentiable functions of their arguments. The relationships that these functions satisfy are given below.

We assume that

$$\omega^\circ(t, x) = \max[\omega_1^\circ(t, x), \omega_2^\circ(t, x)] \tag{3.2}$$

where $w_i^\circ(t, x)$ $(i = 1, 2)$ are given smooth functions. Let

$$a_i(t, x) = \partial\omega_i^\circ(t, x)/\partial t, \ b_i(t, x) = \mathrm{grad}_x \omega_i^\circ(t, x), \ b_1 \neq b_2 \ (i = 1, 2)$$
$$a(t, x) = \lambda a_1 + (1 - \lambda)a_2, \ b(t, x) = \lambda b_1 + (1 - \lambda)b_2$$
$$c(t, x) = \mathrm{grad}_x \Delta\omega^1(t, x), \ d(t, x) = b(t, x) + c(t, x)$$

We assume that the differentiable function $\Delta\omega^1(t, x)$ satisfies the relationships

$$-\partial\Delta\omega^1(t, x)/\partial t = \sup_{\lambda \in [0, 1]} \|d(t, x)\| \varphi(t, x), \ \Delta\omega^1(\theta, x) = 0 \tag{3.3}$$

The differentiable function $\Delta\psi(t)$ satisfies the inequality

$$-d\Delta\psi(t)/dt \geqslant \sup_{(\lambda, x) \in [0,1] \times G} \{\min_{u_1} \max_{v_1} \langle d(t, x), \ A(t)x + B(t)u_1 + C(t)v_1\rangle - \tag{3.4}$$
$$\min_{u_1} \max_{v_1} \langle b(t, x), \ A(t)x + B(t)u_1 + C(t)v_1\rangle\}, \ \Delta\psi(\theta) = 0$$

4. ***Solution of the problem.*** We know /3/ that the upper conjugate derivative of the function $\omega_2^1(t, x)$ is defined by the expression

$$D^*\omega_2^1(t, x)|(s) = \sup_{h \in R^n} (\langle s, h\rangle - \partial_-\omega_2^1(t, x)|(h)) \tag{4.1}$$
$$\partial_-\omega_2^1(t, x)|(h) \overset{\Delta}{=} \liminf_{\delta \downarrow 0} [\omega_2^1(t + \delta, x + \delta h) - \omega_2^1(t, x)]\delta^{-1}$$

where $\partial_- w_2^1(t,x)|(h)$ is the lower derivative of the function $w_2^1(t,x)$ with respect to the direction $(1,h)$.

For any $\omega_2^1(t,x) \in \mathrm{Lip}$, $(t,x) \in I \times G$ the function $h \to \partial_-\omega_2^1(t,x)$ takes finite values and satisfies the Lipschitz condition in the entire space R^n. Since the functions $\Delta\psi(t)$ and $\Delta\omega^1(t,x)$ are differentiable, we will write the upper conjugate derivative $D^*\omega_2^1(t,x)|(s)$ in the form

$$D^*\omega_2^1(t,x)|(s) = \sup_{h\in R^n}(\langle s - c(t,x), h\rangle - \partial_-\omega^\circ(t,x)|(h) - \partial\Delta\omega^1(t,x)/\partial t - d\Delta\psi(t)/dt = D^*\omega^\circ(t,x)|(s - c(t,x)) - \partial\Delta\omega^1(t,x)/\partial t - d\Delta\psi(t)/dt \tag{4.2}$$

The symbol $D^*\omega^\circ(t,x)|(\cdot)$ is the upper conjugate derivative of the function ω° at the point (t,x).

Let us derive an expression for the upper conjugate derivative of the function $\omega^\circ(t,x)$. If the function $\omega^\circ(t,x)$ is differentiable in the position $(t,x) \in I \times G$ in the game (2.1), i.e., the maximum in (3.2) is attained either for $i = 1$ or for $i = 2$, then

$$D^*\omega_i^\circ(t,x)\,|\,(s) = \begin{cases} -\partial\omega_i^\circ(t,x)/\partial t, & s = b_i(t,x) \\ +\infty, & s \neq b_i(t,x) \end{cases}$$

Now suppose that the function $\omega^\circ(t,x)$ is non-differentiable in some position $\{t_*, x_*\}$ Then we have the equality

$$\omega^\circ(t_*, x_*) = \omega_1^\circ(t_*, x_*) = \omega_2^\circ(t_*, x_*)$$

Write the expression for the conjugate derivative in these positons, substituting $s - c(t,x)$ for s. Then

$$D^*\omega^\circ(t,x)\,|\,(s - c(t,x)) = \begin{cases} -a(t,x), & s - c(t,x) = b(t,x) \\ +\infty, & s - c(t,x) \neq b(t,x), \quad 0 \leqslant \lambda \leqslant 1 \end{cases}$$

We will show that the function $\omega_2^1(t,x)$ defined by relationships (3.1)-(3.4) is u-stable for the original non-linear game (1.1), (1.2). To this end, by Theorem 1 of /3/, we have to check the inequality

$$D^*\omega_2^1(t,x)|(s) \geqslant H(t,x,s), \ \forall (t,x,s) \in I \times G \times R^n \tag{4.3}$$

By remark 1, it suffices to check the inequality

$$D^*\omega_2^1(t,x)\,|\,(s) \geqslant H_1(t,x,s) + \varphi(t,x)\|s\|, \quad \forall (t,x,s) \in I \times G \times R^n \tag{4.4}$$

By (4.2), for inequality (4.4) to hold it suffices to have the following inequality:

$$D^*\omega^\circ(t,x)\,|\,(s - c(t,x)) - \partial\Delta\omega^1(t,x)/\partial t - d\Delta\psi(t)/dt \geqslant H_1(t,x,s) + \varphi(t,x)\|s\| \tag{4.5}$$

Let us check inequality (4.5). By assumption, $\omega^\circ(t,x)$ is a u-stable function for the auxiliary linear problem. Therefore, by Theorem 1 of /3/ it satisfies the inequality

$$-a(t,x) \geqslant H_1(t,x,b(t,x)), \quad \forall (t,x) \in I \times G, \quad 0 \leqslant \lambda \leqslant 1 \tag{4.6}$$

By the assumptions of Sect.3, the functions $\Delta\omega^1(t,x)$ and $\Delta\psi(t)$ satisfy the relationships

$$-\partial\Delta\omega^1(t,x)/\partial t = \sup_{\lambda\in[0,1]}\|d(t,x)\|\varphi(t,x), \quad \Delta\omega^1(\theta,x) = 0 \tag{4.7}$$

$$-d\Delta\psi(t)/dt \geqslant \sup_{(\lambda,x)\in[0,1]\times G}[H_1(t,x,d(t,x)) - H_1(t,x,b(t,x)], \ \Delta\psi(\theta) = 0 \tag{4.8}$$

Adding (4.6), (4.7), and (4.8), we obtain

$$\begin{gathered} -a(t,x) - \partial\Delta\omega^1(t,x)/\partial t - d\Delta\psi(t)/dt \geqslant \\ H_1(t,x,b(t,x)) + \sup_{\lambda\in[0,1]}\|d(t,x)\|\varphi(t,x) + \\ \sup_{(\lambda,y)\in[0,1]\times G}[H_1(t,y,d(t,y)) - H_1(t,y,b(\cdot,y)) \geqslant \\ H_1(t,x,d(t,x)) + \sup_{\lambda\in[0,1]}\|d(t,x)\|\varphi(t,x), \ \forall (t,x) \in I \times G, \quad 0 \leqslant \lambda \leqslant 1 \end{gathered}$$

The last inequality implies that the function $\omega_2^1(t,x)$ is u-stable for the original non-linear differential game (1.1), (1.2).

Remark 2. Denote by $\omega^*(t, x)$ the guaranteed value function in problem (1.1), (1.2), which is defined according to the theorem of /13/ by the expression

$$\omega^*(t, x) = \omega^\circ(t, x) + mC^*(\theta - t)$$

where $\omega^\circ(t, x)$ is the value of the auxiliary game (2.1), (2.2), m is the Lipschitz constant with respect to x of the function $\omega^\circ(t, x)$, and the value of the constant C^* is obtained from the expression

$$C^* = \max_{(t, x, s) \in I \times G \times R^n} |H(t, x, s) - H_1(t, x, s)|$$

Thus, if

$$mC^*(\theta - t) \geqslant \Delta\omega^1(t, x) + \Delta\psi(t), \quad \forall (t, x) \in I \times G$$

then

$$\omega_2^1(t, x) \leqslant \omega^*(t, x), \quad \forall (t, x) \in I \times G$$

Remark 3. In the case $\varphi(t, x) \equiv \varphi(t)$, we have $\Delta\psi(t) \equiv 0$.

5. *Example*. Consider the problem of the approach of two objects (pursuers) to a third object (evader). The motion of the pursuers $P_i(y_1^{(i)}, y_2^{(i)})$ is described by the equations

$$\begin{gathered} y_1^{\cdot(i)} = y_3^{(i)}, \ y_3^{\cdot(i)} = -(\alpha + \varepsilon((y_3^{(i)})^2 + (y_4^{(i)})^2)^{1/2} y_3^{(i)} + u_1^{(i)} \\ y_2^{\cdot(i)} = y_4^{(i)}, \ y_4^{\cdot(i)} = -(\alpha + \varepsilon((y_3^{(i)})^2 + (y_4^{(i)})^2)^{1/2} y_4^{(i)} + u_2^{(i)} \\ u^{(i)} = \{0, 0, u_1^{(i)}, u_2^{(i)}\}, \ \ (u_1^{(i)})^2 + (u_2^{(i)})^2 \leqslant \mu^2 > 0. \end{gathered} \tag{5.1}$$

Here $u^{(i)}$ is the control vector of the pursuers, which is selected subject to the above constraints, $\alpha > 0$ is the linear coefficient of viscosity, and $\varepsilon > 0$ is the quadratic coefficient of viscosity (a small quantity). The constraints imposed on the value of ε are given at the end of the paper.

The motion of the evader $E(z_1, z_2)$ is described by the equation

$$z_1^{\cdot} = z_3, \quad z_3^{\cdot} = -\beta z_3^{\cdot} + v_1, \quad z_2^{\cdot} = z_4, \quad z_4^{\cdot} = -\beta z_4 + v_2, \quad v_1^2 + v_2^2 \leqslant \nu^2 > 0 \tag{5.2}$$

We assume that

$$\nu > \mu \tag{5.3}$$

Fix the ending time of the game $t = \theta$. The reward in the game is defined by the continuous function

$$\begin{gathered} \sigma(x) = \min(\sigma_1(x(\theta)), \quad \sigma_2(x(\theta)) \\ \sigma_i(x(\theta)) = [(y_1^{(i)}(\theta) - z_1(\theta))^2 + (y_2^{(i)}(\theta) - z_2(\theta))^2]^{1/2} \quad (i = 1, 2) \end{gathered} \tag{5.4}$$

The argument $x(\theta)$ is the phase vector of system (5.1)-(5.3). From (5.4) it follows that the reward function is non-convex. By (5.4) the reward in the game is the distance between the evader and the nearest pursuer at time $t = \theta$. The strategies of the players and the corresponding motions are defined in accordance with /1, 2/.

The problem is stated as follows: for any initial position of the game (5.1)-(5.4), construct a u-stable guaranteed value function.

According to the proposed approach, we introduce an auxiliary linear differential game with the dynamics of the players described by the equations from /9/. The motion of the pursuers P_i in the auxiliary problem is described by equations corresponding to (5.1) with $\varepsilon \equiv 0$. The evader E in the auxiliary game moves according to Eqs.(5.2), where v is the control vector. The reward in the auxiliary problem is given by (5.4). Note that the controls u and v, as well as the constraints on their values in the auxiliary problem, are identical with the controls u and v and the constraints on their values in the original problem (5.1), (5.2). As in /15/, we assume that in the auxiliary problem

$$\nu \geqslant \mu, \quad \nu / \beta \geqslant \mu / \alpha \tag{5.5}$$

(with at least one of them a strict inequality). The value function of the auxiliary game for all possible positions was derived in /15/. We denote this function by $\omega^\circ(t, x)$ (in /15/ it is denoted by $\varepsilon_0(t, x)$).

We reduce the original system (5.1)-(5.4) and the auxiliary system to a standard-form differential game by the change of variables

$$\begin{gathered} x_1 = y_1^1, \quad x_2 = y_2^1, \quad x_3 = z_1, \quad x_4 = z_2 \\ x_5 = y_3^1, \quad x_6 = y_4^1, \quad x_7 = z_3, \quad x_8 = z_4 \\ x_9 = y_1^2, \quad x_{10} = y_2^2, \quad x_{11} = y_3^2, \quad x_{12} = y_4^2 \end{gathered} \tag{5.6}$$

By (2.3) and (5.6), we have

$$\varphi(t, x) = \max_{s \in S^*} |H(t, x, s) - H_1(t, x, s)| + \varepsilon\varphi_*(t, x) \tag{5.7}$$

where the components of the vector $s \in S^*$ are given by

$$\begin{gathered} s_1 = s_2 = s_3 = s_4 = s_7 = s_8 = s_9 = s_{10} = 0 \\ s_i = x_i \, | y_1^{\cdot} | / \varphi_*(t, x) \; (i = 1, 2), \quad s_j = x_j \, | y_2^{\cdot} | / \varphi_*(t, x) \quad (j = 11, 12) \\ | y_1^{\cdot} | = (x_5^2 + x_6^2)^{1/2}, \quad | y_2^{\cdot} | = (x_{11}^2 + x_{12}^2)^{1/2} \\ \varphi_*(t, x) = (| y_1^{\cdot} |^4 + | y_2^{\cdot} |^4)^{1/2} = [(x_5^2 + x_6^2)^2 + (x_{11}^2 + x_{12}^2)^2]^{1/2} \\ | y_1^{\cdot} |_{\max} = \max_{t \in I} | y_1^{\cdot} |, \quad | y_2^{\cdot} |_{\max} = \max_{t \in I} | y_2^{\cdot} | \end{gathered} \tag{5.8}$$

Recall that in /15/ we identified the phase-space region W in which there is essentially a "two after one" game and showed that the function $\omega^\circ(t, x)$ is expressed in W by the equality

$$\omega^\circ(t, x) = \max(\omega_1^\circ(t, x), \omega_2^\circ(t, x)) \tag{5.9}$$

which, combined with (5.6), gives

$$\begin{gathered} \omega_i^\circ(t, x) = [(D(t, x) - q_i)^2 + F^2(t, x)]^{1/2} - r(t) \\ q_i = x_4 + E_\beta(t) x_8 \pm [R^2(t) - K^2(t, x)]^{1/2} \end{gathered} \tag{5.10}$$

where $i = 1$ corresponds to plus and $i = 2$ to minus. Moreover, from /15/ it follows that

$$\begin{gathered} D(t, x) = x_2 + E_\alpha(t) x_6, \quad F(t, x) = x_1 + E_\beta(t) x_5 \\ K(t, x) = x_3 + E_\beta(t) x_7, \quad R(t) = \nu\beta^{-1}(\theta - t - E_\beta(t)) \\ r(t) = \mu\alpha^{-1}(\theta - t - E_\alpha(t)); \quad E_\gamma(t) = (1 - \exp(-\gamma(\theta - t)))\gamma^{-1}, \quad \gamma = \alpha, \beta \end{gathered} \tag{5.11}$$

By /15/, the functions $\omega_i^\circ(t, x)$ $(i = 1, 2)$, are differentiable for all $(t, x) \in I \times G$.

We can check that on the singular surface S consisting of positions $\{t, x\}$ for which $\omega_1^\circ(t, x) = \omega_2^\circ(t, x)$ we have the equalities

$$\| b_1 \| = \| b_2 \| = \| b_* \| \leqslant [2 + E_\alpha^2(t) + E_\beta^2(t)]^{1/2} \tag{5.12}$$

Using (5.12) and (5.11), we obtain

$$\sup_{\lambda \in [0, 1]} \| \lambda b_1 + (1 - \lambda) b_2 \| = \| b_* \|, \quad \{t, x\} \in S \tag{5.13}$$

The guaranteed value function in this example is defined by expression (3.1), in which $\omega^\circ(t, x)$ is given by (5.10), (5.11). Eq.(3.3) for this examples preserves its form, with $\varphi(t, x)$ defined by (5.7), (1.8).

We can show that to within $o(\varepsilon)$ the function $\Delta\omega^1(t, x)$, defined by

$$\Delta\omega^1(t, x) = \varepsilon\varphi_*(x) B(t, \theta), \; B(t, \theta) = \int_t^\theta \| b(\tau) \| \, d\tau \tag{5.14}$$

is a solution of Eq.(3.3).

Consider the function $\Delta\omega_*^1(t, x)$ which is a solution of the equation

$$\begin{gathered} -\partial\Delta\omega_*^1(t, x) / \partial t = \varepsilon\varphi_*(x)(\| b_*(t) \| + \varepsilon c(t, x) B(t, \theta)) \\ \Delta\omega_*^1(\theta, x) = 0 \end{gathered}$$

where the right-hand side majorizes the right-hand side of Eq.(3.3) with $\Delta\omega^1(t, x)$ substituted from (5.14). We can show that the inequality

$$\begin{gathered} \Delta\omega^1(t, x) \leqslant \Delta\omega_*^1(t, x) = \varepsilon\varphi_*(x)(B(t, \theta) + \varepsilon c(t, x) \Lambda(t)) \\ \Lambda(t) = \int_t^\theta B(\tau, \theta) \, d\tau \end{gathered} \tag{5.15}$$

holds for all $(t, x) \in I \times G$.

The function $\Delta\psi(t)$ must satisfy Eq.(3.4), which for this example takes the form

$$\begin{gathered} -d\Delta\psi(t) / dt \geqslant \sup_{(\lambda, x) \in [0, 1] \times G} \{-\alpha(\Delta s_5 s_5 + \Delta s_6 s_6 + \Delta s_{11} s_{11} + \Delta s_{12} s_{12}) + \\ \min_{u^{(1)}} [(s_5 + \Delta s_5) u_1^{(1)} + ((2\lambda - 1) s_6 + \end{gathered} \tag{5.16}$$

$$\Delta s_6)\, u_2^{(1)}] - \min_{u^{(1)}} [s_5 u_1^{(1)} + (2\lambda - 1)\, s_6 u_2^{(1)}] + \min_{u^{(2)}} [\Delta s_{11} u_1^{(2)} + \Delta s_{12} u_2^{(2)}]\}$$

Here

$$s_5 = \partial \Delta\omega_1^\circ (t, x) / \partial x_5 = F(t, x)\, E_\alpha (t) / [R^2 (t) - K^2 (t, x) + F^2 (t, x)]^{1/2} \tag{5.17}$$

$$s_6 = \partial \Delta\omega_1^\circ (t, x) / \partial x_6 = -[R^2 (t) - K^2 (t, x)]\, E_\alpha (t) / [R^2 (t) - K^2 (t, x) + F^2 (t, x)]^{1/2}$$

$$\Delta s_i = \partial \Delta\omega^1 (t, x) / \partial x_i = 2\varepsilon B (t, \theta) [\, |y_1^\cdot|^2 x_i / \varphi_* (t, x)] \quad (i = 5, 6)$$

$$\Delta_j = \partial \Delta\omega^1 (t, x) / \partial x_j = 2\varepsilon B (t, \theta) [\, |y_2^\cdot|^2 x_j / \varphi_* (t, x)] \quad (j = 11, 12)$$

Making the required transformations in (5.16), substituting $\Delta s_5, \Delta s_6, \Delta s_{11}, \Delta s_{12}, s_5, s_6$ from (5.17), and changing the inequality to an equality, we obtain an equation for $\Delta\psi(t)$:

$$d\Delta\psi (t) / dt = -2\varepsilon B (t, \theta) \max_{y_1^\cdot} \psi (y_1^\cdot), \quad \Delta\psi (\theta) = 0 \tag{5.18}$$

$$\psi (y_1^\cdot) = -\alpha (y_1^\cdot)^2 + \mu y_1^\cdot$$

We can show that

$$\max_{y_1^\cdot} \psi (y_1^\cdot) = \begin{cases} -\alpha\, |y_1^\cdot|_{\max}^2 + \mu\, |y_1^\cdot|_{\max}, & |y_1^\cdot|_{\max} < \mu/(2\alpha) \\ \mu^2/(4\alpha), & |y_1^\cdot|_{\max} \geqslant \mu/(2\alpha) \end{cases} \tag{5.19}$$

From (5.18), using (5.19), we obtain

$$\Delta\psi (t) = 2\varepsilon \max_{y_1^\cdot} \psi (y_1^\cdot)\, \Lambda (t) \tag{5.20}$$

Using (5.8), (5.9), (5.10), (5.15), and (5.20), we obtain a final bound on the guaranteed value in this example:

$$\omega_2^1 (t, x) \leqslant \omega_{2*}^1 (t, x) = \omega^\circ (t, x) + \varepsilon\varphi_* (x) (B (t, \theta) + \varepsilon c (t, x)\, \Lambda (t)) + 2\varepsilon \max_{y_1^\cdot} \psi (y_1^\cdot)\, \Lambda (t) \tag{5.21}$$

Remark 4. Let us compare the value (5.21) with the guaranteed value $\omega^* (t, x)$ for this example obtained according to /13/:

$$\omega^* (t, x) = \omega^\circ (t, x) + C^* m (\theta - t), \quad C^* = \varepsilon \sqrt{2}\mu^2/\alpha^2 \tag{5.22}$$

$$m = \max_{t \in I} \| b_* (t) \| = [2 + E_\alpha^2 (t_0) + E_\beta^2 (t_0)]^{1/2}$$

$\omega^\circ (t, x)$ is defined in (5.9), (5.10), and m is the Lipschitz constant of the function $\omega^\circ (t, x)$ defined by (5.7).

To compare (5.21) and (5.22), we take initial conditions that are the "worst" for the value (5.21), assuming that (x_5^0, x_6^0) and (x_{11}^0, x_{12}^0) satisfy the equalities

$$[(x_5^0)^2 + (x_6^0)^2]^{1/2} = \mu/\alpha, \quad [(x_{11}^0)^2 + (x_{12}^0)^2]^{1/2} = \mu/\alpha \tag{5.23}$$

The other initial coordinates of the players P_i $(i = 1, 2)$ and E do not affect the difference of the values. We also assume that

$$\max_{y_1^\cdot} \psi (y_1^\cdot) = \mu^2/(4\alpha) \tag{5.24}$$

Substituting C^* from (5.22) and the initial conditions (5.23) into (5.21), we have

$$\omega_2^1 (y_1^\cdot) \leqslant \omega_{2*}^1 (t_0, x_0) = w^\circ (t_0, x_0) + \varepsilon \sqrt{2}\mu^2\alpha^{-2} B (t_0, \theta) + \varepsilon^2 \sqrt{2}\mu^3\alpha^{-3} \Lambda (t_0) + \varepsilon\mu^2 (2\alpha)^{-1} \Lambda (t_0) \tag{5.25}$$

From (5.25) and (5.22) it follows that the value (5.25) obtained by our technique is less than the value (5.22) obtained by the technique of /13/ if

$$\alpha\mu^{-1} [m (\theta - t_0) - B (t_0, \theta) + \alpha (2\sqrt{2})^{-1} \Lambda (t_0)] \geqslant \varepsilon\Lambda (t_0) \tag{5.26}$$

Note that the expression in brackets on the left-hand side of (5.26) can be shown to be non-negative for all α satisfying the inequality $0 \leqslant \alpha \leqslant \alpha_0$, where $\alpha_0 > 0$ is a constant.

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